Analytical r-mode solution with gravitational radiation reaction force

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We present and discuss the analytical r-mode solution to the linearized hydrodynamic equations of a slowly rotating, Newtonian, barotropic, non-magnetized, perfect-fluid star in which the gravitational radiation reaction force is present.

I. INTRODUCTION

The r-modes, which are pulsation modes of rotating stars that have the Coriolis force as their restoring force, are driven unstable by gravitational radiation (GR) [1]. In the frame co-rotating with the star, the energy of the unstable r-mode grows exponentially, $E = E_0 e^{-2t/\tau_{GR}}$, with the gravitational timescale τ_{GR} given by [2, 3]

$$\tau_{\mathrm{GR}} = - \left(\frac{2^{17}\pi}{5^2 \times 3^8} \frac{G}{c^7} \tilde{J}\Omega^6 \right)^{-1}, \qquad (1)$$

where G is the Newton's constant, c is the velocity of light, Ω is the angular velocity of the star and $\tilde{J} \equiv \int_0^R dr\hat{\rho}r^6$ with $\hat{\rho}$ and R being, respectively, the density and the surface's radius of the unperturbed star. For a wide range of relevant temperatures and angular velocities of newly born, hot, rapidly-rotating stars, bulk and shear viscosity do not suppress the exponential growth of the energy of the r-mode [2, 3].

In the above-mentioned investigations [2, 3], the linearized hydrodynamics equations with the GR force were used to obtain an expression for the time evolution of the physical energy of the r-mode perturbation, dE/dt, from which the gravitational radiation and viscous timescales were determined. In this work, in which the main results of Ref. [4] are reported, we present an explicit expression for the r-mode velocity perturbations that solves the linearized hydrodynamics equations with the GR force. Our conclusions are that (i) these velocity perturbations are sinusoidal with the same frequency as the well-known GR force-free linear r-mode solution, (ii) the GR force drives the r-modes unstable with a growth timescale that agrees with the expression found in Refs. [2, 3], and (iii) the amplitude of these velocity perturbations is corrected, relatively to the GR force-free case, by a term of order Ω^6.

II. HYDRODYNAMIC EQUATIONS WITH GR REACTION FORCE

The Newtonian hydrodynamic equations for a uniformly rotating, barotropic, non-magnetized, perfect-fluid star in the presence of the gravitational radiation (GR) reaction force are the Euler, continuity, and Poisson equations given, respectively, by

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\rho^{-1}\vec{\nabla}P - \vec{\nabla}\Phi + \vec{F}^{\mathrm{GR}}, \qquad (2)$$

$$\partial_t \rho + \vec{\nabla} \cdot (\rho\vec{v}) = 0, \qquad (3)$$

$$\nabla^2\Phi = 4\pi G\rho, \qquad (4)$$

where ρ, P and $\vec{v}$ are, respectively, the density, the pressure and the velocity of the fluid, Φ is the Newtonian potential, and the GR reaction force,

$$\vec{F}^{\mathrm{GR}} = -\partial_t\vec{\beta} + \vec{v} \times (\vec{\nabla} \times \vec{\beta}), \qquad (5)$$

is assumed to be given by the 3.5 post-Newtonian order expansion that includes the contribution of the current quadrupole moment, which is the main responsible for the GR instability that sets in [5, 6]. In Eq. (5), $\vec{\beta}$ is the gravitational vector potencial whose components are given by

$$\beta_i \equiv \frac{16G}{45c^7} \epsilon_{ijk}x_j x_q S_{kq}^{[5]}, \qquad (6)$$

where $S_{ij}(t)$ is the time-varying current quadrupole tensor,

$$S_{ij}(t) \equiv \int d^3x\epsilon_{kq(i}x_{j)}x_k\rho v_q, \qquad (7)$$

ϵ_{ijk} is the Levi-Civita tensor, x_i is the Cartesian coordinate of the point at which the tensor is evaluated, and $S_{ij}^{[n]}(t)$ denotes the nth time derivative of S_{ij}.

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III. LINEARIZED HYDRODYNAMICS EQUATIONS WITH GR REACTION FORCE

The hydrodynamic equations (2)–(4) can be linearized, yielding

$$\partial_t \delta^{(1)}\vec{v} + (\delta^{(1)}\vec{v} \cdot \vec{\nabla})\hat{\vec{v}} + (\hat{\vec{v}} \cdot \vec{\nabla})\delta^{(1)}\vec{v}$$
$$= -\vec{\nabla}\delta^{(1)}U + \delta^{(1)}\vec{F}^{\mathrm{GR}}, \tag{8}$$

$$\partial_t \delta^{(1)}\rho + \hat{\vec{v}} \cdot \vec{\nabla}\delta^{(1)}\rho + \vec{\nabla} \cdot \left(\hat{\rho}\delta^{(1)}\vec{v}\right) = 0, \tag{9}$$

$$\nabla^2 \delta^{(1)}\Phi = 4\pi G \delta^{(1)}\rho, \tag{10}$$

where $\hat{\vec{v}} = \Omega r \sin\theta \vec{e}_\phi$ is the velocity of the unperturbed star, $\hat{\rho}$ its mass density, $\delta^{(1)}Q$ denotes the first-order Eulerian change in a quantity Q and we have defined $\delta^{(1)}U \equiv \delta^{(1)}P/\hat{\rho} + \delta^{(1)}\Phi$.

To compute the first-order Eulerian change in the GR force, $\delta^{(1)}\vec{F}^{\mathrm{GR}}$, we assume that the GR force-free r-mode velocity perturbations[1],

$$\delta^{(1)}v_r = 0, \tag{11a}$$

$$\delta^{(1)}v_\theta = -\frac{i}{4}\sqrt{\frac{5}{\pi}}\frac{\alpha\Omega}{R}r^2 \sin\theta e^{i(2\phi+\omega t)}, \tag{11b}$$

$$\delta^{(1)}v_\phi = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{\alpha\Omega}{R}r^2 \sin\theta \cos\theta e^{i(2\phi+\omega t)}, \tag{11c}$$

act as a source for the first-order Eulerian change in the current quadrupole tensor, $\delta^{(1)}S_{ij}$. In the above expression, α is the amplitude of the r-mode and we assume that

$$\omega = \omega_0 + i\varpi, \tag{12}$$

where the frequency of the r-mode, $\omega_0 \equiv \mathrm{Re}[\omega]$, and the small imaginary part that is related to the growth timescale of the instability of the mode, $\varpi = \mathrm{Im}[\omega] < 0$, are arbitrary parameters to be determined.

Under the above assumptions, $\delta^{(1)}S_{xx}$ is computed to be

$$\delta^{(1)}S_{xx} = -\alpha\Omega\sqrt{\frac{\pi}{5}}\frac{\tilde{J}}{R}e^{-\varpi t}e^{i\omega_0 t}, \tag{13}$$

where we have neglected the contribution coming from the terms $\hat{v}_i\delta^{(1)}\rho$ (of order $\alpha\Omega^3$) and retained only the dominant terms $\hat{\rho}\delta^{(1)}v_i$ (of order $\alpha\Omega$). Similarly, it is straightforward to show that the first-order Eulerian change in the other components of the quadrupole tensor satisfy the relations

$$\delta^{(1)}S_{yy} = -\delta^{(1)}S_{xx},$$
$$\delta^{(1)}S_{xy} = i\delta^{(1)}S_{xx},$$
$$\delta^{(1)}S_{xz} = \delta^{(1)}S_{yz} = \delta^{(1)}S_{zz} = 0. \tag{14}$$

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[1] In this article, we will be concerned exclusively with the $l = 2$ r-mode, which is the most unstable mode.

Using Eqs. (13) and (14), the first-order Eulerian change in the gravitational vector potential, $\delta^{(1)}\vec{\beta}$, is then computed to be

$$\delta^{(1)}\beta_r = 0, \tag{15a}$$

$$\delta^{(1)}\beta_\theta = -\kappa\frac{\tilde{J}}{R}\alpha\Omega\omega_0^5 r^2 \sin\theta e^{-\varpi t}e^{i(2\phi+\omega_0 t)}, \tag{15b}$$

$$\delta^{(1)}\beta_\phi = -i\kappa\frac{\tilde{J}}{R}\alpha\Omega\omega_0^5 r^2 \sin\theta \cos\theta e^{-\varpi t}e^{i(2\phi+\omega_0 t)}, \tag{15c}$$

where the constant κ sets the strength of the GR reaction force and is defined as

$$\kappa \equiv \frac{16}{45}\sqrt{\frac{\pi}{5}}\frac{G}{c^7}. \tag{16}$$

Finally, taking into account that $\hat{S}_{ij} = 0$, implying that $\hat{\beta}_i = 0$, the first-order Eulerian change in the GR force, $\delta^{(1)}\vec{F}^{\mathrm{GR}}$, is computed to be

$$\delta^{(1)}F_r^{\mathrm{GR}} = -3i\kappa\frac{\tilde{J}}{R}\alpha\Omega^2\omega_0^5 r^2 \sin^2\theta \cos\theta$$
$$\times e^{-\varpi t}e^{i(2\phi+\omega_0 t)}, \tag{17a}$$

$$\delta^{(1)}F_\theta^{\mathrm{GR}} = i\kappa\frac{\tilde{J}}{R}\alpha\Omega\omega_0^5 \left(\omega_0 + 3\Omega\sin^2\theta\right) r^2 \sin\theta$$
$$\times e^{-\varpi t}e^{i(2\phi+\omega_0 t)}, \tag{17b}$$

$$\delta^{(1)}F_\phi^{\mathrm{GR}} = -\kappa\frac{\tilde{J}}{R}\alpha\Omega\omega_0^6 r^2 \sin\theta \cos\theta$$
$$\times e^{-\varpi t}e^{i(2\phi+\omega_0 t)}. \tag{17c}$$

IV. THE ANALYTICAL R-MODE SOLUTION WITH GR REACTION FORCE

The linearized hydrodynamic equations (8)–(10), with the first-order Eulerian change in the GR force, $\delta^{(1)}\vec{F}^{\mathrm{GR}}$, given by Eq. (17), admit the solution [4]

$$\delta^{(1)}v_r = 0, \tag{18a}$$

$$\delta^{(1)}v_\theta = -\frac{i}{2}\alpha\Omega\left[\frac{1}{2}\sqrt{\frac{5}{\pi}}\frac{1}{R} + i\gamma(A-1)\Omega^5\right]r^2 \sin\theta$$
$$\times e^{-\varpi t}e^{i(2\phi+\omega_0 t)}, \tag{18b}$$

$$\delta^{(1)}v_\phi = \frac{1}{2}\alpha\Omega\left[\frac{1}{2}\sqrt{\frac{5}{\pi}}\frac{1}{R} + i\gamma(A-1)\Omega^5\right]r^2 \sin\theta \cos\theta$$
$$\times e^{-\varpi t}e^{i(2\phi+\omega_0 t)}, \tag{18c}$$

and

$$\delta^{(1)}U = \frac{1}{3}\alpha\Omega^2\left(\frac{1}{2}\sqrt{\frac{5}{\pi}}\frac{1}{R} + i\gamma A\Omega^5\right)r^3 \sin^2\theta \cos\theta$$
$$\times e^{-\varpi t}e^{i(2\phi+\omega_0 t)}, \tag{19}$$

with ϖ and ω_0 given by

$$\varpi = -\frac{8}{9}\sqrt{\frac{\pi}{5}}\gamma R\Omega^6 \quad \text{and} \quad \omega_0 = -\frac{4\Omega}{3}. \tag{20}$$

The velocity perturbations given by Eq. (18) have a piece similar to the GR force-free solution, the difference being the factor $e^{-\varpi t}$ responsible for the exponential growth of the r-mode amplitude due to the presence of a GR reaction force, and another piece proportional to $\alpha\gamma(A-1)\Omega^6$, where A is a constant fixed by the choice of initial data and $\gamma \equiv 1024\kappa\tilde{J}/(81R)$.

The GR force-free limit is obtained when we set the parameter κ, defined in Eq. (16), equal to zero. In this limit, ϖ and γ also go to zero. Then, from Eqs. (18) and (19), we recover the GR force-free linear r-mode solution.

V. DISCUSSION OF THE RESULTS AND FUTURE DIRECTIONS

In this work we have presented the analytical r-mode solution to the linearized Newtonian hydrodynamic equations with the GR reaction force. The velocity perturbations $\delta^{(1)}\vec{v}$ are proportional to $e^{i(2\phi+\omega_0 t)}$, with $\omega_0 = -4\Omega/3$. Thus, they have the same sinusoidal behavior and the same frequency ω_0 as the GR force-free velocity perturbations given by Eq. (11). The amplitude of the velocity perturbations is proportional to $\exp\{-\varpi t\}$. Since $\varpi < 0$, the GR force induces then an exponential growth in the r-mode amplitude. The e-folding growth timescale $\tau_{GR} = 1/\varpi$ agrees with the GR timescale (1) found in Refs. [2, 3]. The velocity perturbations $\delta^{(1)}\vec{v}$ contain also a piece proportional to $\alpha\gamma(A-1)\Omega^6$, where A is an arbitrary constant fixed by the choice of initial data. If we choose this constant A to be of order unity, then this part of the solution could be neglected [4].

A quite interesting feature that has emerged from recent investigations on r-modes is the presence of differential rotation induced by the r-mode oscillation in a background star that is initially uniformly rotating. That differential rotation drifts of kinematical nature could be induced by r-mode oscillations of the stellar fluid was first suggested in Ref. [7]. The existence of these drifts was confirmed in numerical simulations of nonlinear r-modes carried out both in general relativistic hydrodynamics [8] and in Newtonian hydrodynamics [9]. Differential rotation was also reported in a model of a thin spherical shell of a rotating incompressible fluid [10]. Recently, an analytical solution, representing differential rotation of r-modes that produce large scale drifts of fluid elements along stellar latitudes, was found within the nonlinear Newtonian theory up to second order in the mode amplitude and in the absence of GR reaction [11]. This differential rotation plays a relevant role in the nonlinear evolution of the r-mode instability [12]. Two questions could be then naturally raised, namely, is differential rotation also induced by the gravitational radiation reaction and does this differential rotation play a relevant role in the nonlinear evolution of the r-mode instability? One of the aims of the investigation carried out in Ref. [4] is to initiate a programme that hopefully will allow to answer this question. The natural continuation of this investigation is then to try to find an analytical r-mode solution of the *nonlinear* hydrodynamic equations with the GR reaction force. This work is in progress [13].

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